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Bounds on Malapportionment

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ABSTRACT

Uniformly sized constituencies give voters similar influence on election outcomes. When constituencies are set up, seats are allocated to the administrative units, such as states or counties, using apportionment methods. According to the impossibility result of Balinski and Young, none of the methods satisfying basic monotonicity properties assign a rounded proportional number of seats (the Hare-quota). We study the malapportionment of constituencies and provide a simple bound as a function of the house size for an important class of divisor methods, a popular, monotonic family of techniques.

1. Introduction

In most democratic countries, some or all members of the Parliament are elected directly by the voters in electoral districts or (single-member) constituencies. For practical considerations these constituencies are embedded in the countries' existing administrative units, such as states or counties. To ensure equal representation, states are allotted seats in proportion to their populations. As fractional seats cannot be allocated, a fair division problem ensues. This is the so-called *apportionment problem*. Given an apportionment, the constituency boundaries can be designed in each region. This is also a non-trivial task as small towns cannot be split into two parts belonging to different constituencies. Thus, *districting* also makes proportional representation more difficult.

Proportional representation is not always pursued as a goal for all institutions (e.g. European Parliament, US Senate). Furthermore, some countries deliberately stray from proportional distribution to strengthen the representation of rural areas (e.g. Spain). Nevertheless, proportionality remains the fundamental principle of apportionment.

The 14th Amendment of the US Constitution already established that proportionality should be the key factor in apportionment. Since then, the US Supreme Court repeatedly confirmed that no deviation from equality is too small to challenge as long as a plan with less inequality can be presented (see the case Kirkpatrick v. Preisler (1969)). In Europe the Venice Commission, the advisory body of the Council of Europe in the field of constitutional law, published a guidebook for drafting electoral laws. The Code of Good Practice in Electoral Matters also attested that equality of voting power should be achieved by creating constituencies of equal size ([13], §13–15 in Section 2.2).

Even if the constituencies can be equalized within a state, there will be some deviations across states due to divisibility issues. The cited Supreme Court decision ordered the state of Missouri to redesign the districts because the attained 0.69% difference was not the lowest possible. In contrast, the constituencies of Montana are 88% larger than those of Rhode Island [2]. How much of this discrepancy is inherent? Is it possible to significantly decrease this gap? We aim to answer this question in this short note. We focus on apportionment, and disregard the difficulties that arise with the actual design of constituencies.

The Venice Commission itself advises that the gap should not be larger than 10% or, under exceptional circumstances, 15%. Since this requirement is hard to meet, many countries (including France, Germany, and Hungary) use a more relaxed interpretation: difference is measured from the average constituency size rather than pairwise. Indeed, the first draft of Hungary's redesigned electoral law in 2011 based on the stricter rule was mathematically impossible to satisfy. In the final version it was changed to the more relaxed interpretation.

What are feasible differences in general? We look at mainstream apportionment methods, establish bounds on the maximum of this difference as a function of the house size, and illustrate our results by data from Norway. Finally we note that the Impossibility Theorem of Balinski and Young [1] can often be resolved: certain methods, such as the Sainte-Laguë/Webster

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method, almost always satisfy the requirements, otherwise the Hare-quota requirement could be replaced by a softer condition as recommended by the Venice Commission.

2. Apportionment methods

We define the apportionment problem and methods. Let $N = \{1, 2, \ldots, n\}$ be the set of states of the country. An *apportionment problem* $(\boldsymbol{p}, H)$ is a pair consisting of a vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ of state populations $p_i \in \mathbb{N}_+$ and a positive integer $H \in \mathbb{N}_+$ denoting the number of seats in the house. An *apportionment method* determines the non-negative integers $a_1, a_2, \ldots, a_n$ with $\sum_{i=1}^n a_i = H$, specifying how many constituencies each of the states $1, 2, \ldots, n$ gets. Formally, it is a function M that assigns an allotment for each apportionment problem $(\boldsymbol{p}, H)$. Let $P = \sum_{i=1}^n p_i$ be the population of the country, and let $A = \frac{P}{H}$ denote the average size of a constituency. The fraction $\frac{p_i}{P}H = \frac{p_i}{A}$ is the *respective share* of state i.

Rounding the respective shares up or down is a natural way to obtain an apportionment. Apportionment methods that produce allotments by some form of rounding are said to exhibit the *Hare-quota*, or simply *quota* property. *Largest remainder methods* were explicitly designed with this property in mind. The most widely known method is the Hamilton-method, which first assigns the lower integer part of its respective share, the so-called *lower quota*, to each state, and then the remaining seats are distributed one-by-one to the states with the largest fractional parts of their respective shares.

Divisor methods constitute another family of apportionment techniques. An apportionment method is a *divisor method* if there exists a monotone increasing function $f : \mathbb{N} \to \mathbb{R}$, the *divisor criterion*, such that the seats are allocated to the state with the highest $\frac{p_i}{f(s)}$ value in each round. More precisely, suppose that $k-1$ seats are already allotted and the resulting apportionment is $\boldsymbol{a}$, then the kth seat goes to the state for which the fraction $\frac{p_i}{f(a_i)}$ is the highest. We assume that all of the $\frac{p_i}{f(a_i)}$ values are distinct. Ties are unlikely, for real data no tie-breaking rules are specified. The $\frac{p_i}{f(s)}$ value is the *rank-index* or *claim* of state i when it has s seats. Common divisor methods include the following (EP stands for Equal Proportions method — aliases are due to reinventions):

Adams method	$f(s) = s$
Danish method	$f(s) = s + 1/3$
Harmonic mean/Dean method	$f(s) = \dfrac{2s(s+1)}{2s+1}$
Huntington-Hill/EP method	$f(s) = \sqrt{s(s+1)}$
Sainte-Laguë/Webster method	$f(s) = s + 1/2$
Jefferson/D'Hondt method	$f(s) = s + 1$

The divisor criteria are listed in pointwise increasing order from Adams to Jefferson/D'Hondt; the methods favour large states over small states in the same order. That is, the Adams method favours small states the most, while the Jefferson/D'Hondt is the most beneficial for large states (see also Theorem 4 and [1,7,8]). The principal advantage of divisor methods is their immunity to paradoxes related to monotonicity, such as the Alabama-paradox.

We call divisor methods with $s \le f(s) \le s + 1$ *regular divisor methods*. More exotic divisor methods like the Imperiali ($f(s) = s + 2$) or the Macau ($f(s) = 2^s$) methods are not regular. Interestingly, while the Imperiali-method favours large states even more than the Jefferson/D'Hondt, the Macau-method is drastically small-state-friendly. Hence, it is false to conclude that the larger the divisor, the better it is for the large states.

The class of regular divisor methods is larger than it seems. The distribution of seats only depends on the relative order of claims, which does not change if all the claims are multiplied with the same (positive) number.

Remark 1. For any μ, ν such that $\frac{\nu}{\mu} \le 1$ the divisor method with $f(s) = \mu s + \nu$ is regular and equivalent with the divisor method with $\hat{f}(s) = s + \nu/\mu$.

This explains, why the Sainte-Laguë/Webster method is sometimes defined with $f(s) = 2s + 1$.

A third branch of apportionment methods aims to minimize the range of populations. The *minimum range method* [3,4] minimizes the maximum disparity in representation between any two states, while the *Leximin* method [2], lexicographically minimizes the maximum departure, that is, the difference between the population of any constituency and the average constituency size.

Malapportionment measures have been studied by [6,10,11,14]. We look at departure from the average constituency size as a more explicit and intuitive measure of malapportionment.

3. Departure as a malapportionment measure

Let the relative difference displayed by the constituencies of state i be denoted by

$$\delta_i = \frac{\frac{p_i}{a_i} - A}{A},$$

and let $d_i = |\delta_i|$ be the *departure*, its absolute value. *Maximum departure* of an allotment, $\mathbf{a} = (a_1, \ldots, a_n)$ is the maximum of the d_i values for $i = 1, 2, \ldots, n$.

Let $l_i = \left\lfloor \frac{p_i}{P}H \right\rfloor$ and $u_i = \left\lceil \frac{p_i}{P}H \right\rceil$ denote the lower and upper quotas of state i, respectively, and let β_i (for *best* case) and ω_i (for *worst* case) denote the minimum and maximum differences achievable for state i when it gets the lower or upper integer part of its respective share.

$$\beta_i = \min\left(\left| \frac{\frac{p_i}{l_i} - A}{A} \right|, \left| \frac{\frac{p_i}{u_i} - A}{A} \right| \right), \qquad \beta = \max_{i \in N} \beta_i,$$

$$\omega_i = \max\left(\left| \frac{\frac{p_i}{l_i} - A}{A} \right|, \left| \frac{\frac{p_i}{u_i} - A}{A} \right| \right), \qquad \omega = \max_{i \in N} \omega_i.$$

Here β, the maximum of the β_i values, is a natural, not necessarily tight lower bound on the maximum departure for any apportionment. Similarly, the maximum of the ω_i values, denoted by ω, is an upper bound for any apportionment *which satisfies the Hare-quota*. If an apportionment does not satisfy Hare-quota, then it may have a departure larger than ω.

The β and ω bounds indicate that proportional representation relies on our ability to round the critical states in a good direction. Unfortunately, keeping the total at H forces us to allocate seats suboptimally. Suppose that there are seats left after an optimal rounding: Which state should we give them to? Should each state get only one extra seat (rounding it up rather than down as it would optimal)? Rounding in the wrong direction may increase departure drastically for small states, while for larger states even adding multiple seats has a minor effect on the relative difference, that is, departure, making such states ideal buffers to store seats that would mess up the apportionment. A similar argument applies to the case when the optimal allocation would distribute too many seats.

Enforcing quota ensures that the departure will not exceed ω, but the additional constraint also makes it difficult to stay close to β, since we cannot use states as buffers to lend/borrow problematic or desperately needed seats for critical states without creating too much inequality. What are these critical states? They are small states, which are only a few times the size of the average

Table 1
Quota violations by regular and non-regular divisor methods (in boldface). Note that for regular-methods the violation appears only at the largest state.

		Population		
		810	1000	8200
$H = 33$	Respective share	2.67	3.29	27.03
	Adams	3	4	**26**
H= 35	Respective share	2.83	3.39	28.67
	Jefferson/D'Hondt	2	3	**30**
	Imperiali	2	**2**	**31**
	Macau	**10**	**11**	**14**

constituency. It is easy to prove the following upper bounds:

$$\beta \leq \overline{\beta} \stackrel{def}{=} \frac{1}{2l_{sm}+1},$$

$$\omega \leq \overline{\omega} \stackrel{def}{=} \begin{cases} \frac{1}{l_{sm}} & \text{if } l_{sm} > 0, \\ \infty & \text{if } l_{sm} = 0; \end{cases}$$

where l_{sm} denotes the lower integer part of the smallest state's respective share [5].

4. Examples

Both the minimum range and the Leximin methods tends to treat large states as buffers [3,2]: large states may get more seats than their upper quota or less seats than their lower quota in order to balance out the small critical states. This is not surprising at all, considering that the average constituency size in large states is less affected if the allotted number of seats changes. Curiously enough, this can be observed for divisor methods too.

Consider a country with three states (Table 1). When the house size is fixed at 33, the Adams method, which favours the small states, gives state 1 and 2 their upper quotas. To pull this off it gives the largest state, state 3, one less seats than its lower quota. In contrast, when $H = 35$, the Jefferson/D'Hondt method gives the small states their lower quota and, to account for the excess seat, it gives the largest state more than its upper quota. In the next section, Theorem 6 states that if a quota violation happens, there will always be a smaller state where no violation appears. Empirical data shows that for regular divisor methods, quota violations happen only for some of the largest states. However, this is no longer true for the Imperiali- or the Macau-method (Table 1).

Quota violations are more common for Adams and the Jefferson/D'Hondt methods than for the Huntington–Hill/EP and the Sainte-Lagüe/Webster methods [5,12]. Nevertheless all divisor methods violate quota from time to time.

Fig. 1 depicts the maximum departure produced by various methods on Norwegian data under different house sizes. To keep the figure transparent we only sketched four solutions. The Leximin method, devised to minimize departure, coincides with β in this case. We computed other regular divisor methods (like the Danish and the EP methods) too, and none of them violated the ω bound. Is this true in general? Empirical evidence from a number of countries suggests that *regular* divisor methods never produce a maximum departure that exceeds this bound. In the next section, we will prove this observation for regular divisor methods that have a *linear* divisor criterion.

5. Bounds for regular divisor methods

Regular divisor methods with a linear divisor criterion include the Adams, Danish, Sainte-Lagüe/Webster, and Jefferson/D'Hondt methods. In this section we prove that no such method violates the ω bound. It is a well-known fact that divisor methods violate the Hare-quota property; regular divisor methods cannot violate both upper and lower quota at the same time [9, Section 11.4].

Lemma 2. *For any regular divisor method and apportionment problem $(\boldsymbol{p}, H)$, if there exists a state that received more seats that its upper quota then each state received at least its lower quota. Conversely, if there exists a state which received less seats that its lower quota then each state received at most its upper quota.*

Proof. Suppose that state i received k seats, which is strictly more than its upper quota. By contradiction, suppose there is a state, j that received m seats, which is less than its lower quota. Let $\ell \leq m$ be the number of seats j had when i received its kth seat. Then

$$A \geq \frac{p_i}{u_i} \geq \frac{p_i}{f(k-1)} > \frac{p_j}{f(\ell)} \geq \frac{p_j}{f(m)} \geq \frac{p_j}{l_j} \geq A$$

which is a contradiction. The average constituency size A lies between the upper and the lower quotas, hence the first and last inequalities are trivial. The second follows from $u_i \leq k-1 \leq f(k-1)$ and the (strict) third one from that M incremented i and not j when i had $k-1$ seat and j had ℓ. The fourth inequality follows from the choice of ℓ and m and the monotonicity of f. Finally, $f(m) \leq m+1 \leq l_j$. □

Lemma 3. *Lemma 2 extends to the Leximin method.*

Proof. Assume the contrary, and consider a transfer of seats between the violators on different sides. The result is a lexicographic improvement. □

For the sake of completeness, we prove a well-known property of the Adams and the Jefferson/D'Hondt methods. Namely, that they only violate the Hare-quote in one direction [9, Section 11.4].

Theorem 4. *The Jefferson/D'Hondt method may only violate the upper, the Adams method only the lower quota.*

Proof. For the Jefferson/D'Hondt method, $f(s) = s+1$. By contradiction, suppose that state i receives less seats than its lower quota. Then, there must be a state j which received its upper quota (and $l_j \neq u_j$). That is at one point state i had $\ell < l_i$ seats, state j had l_j seats and the Jefferson method incremented state j, which implies that $\frac{p_i}{f(\ell)} < \frac{p_j}{f(l_j)}$. However this leads to a contradiction as

$$A \leq \frac{p_i}{l_i} \leq \frac{p_i}{\ell+1} = \frac{p_i}{f(\ell)} < \frac{p_j}{f(l_j)} = \frac{p_j}{u_j} < A.$$

Now let us fix $f(s) = s$. By contradiction suppose that state i receives more seats than its upper quota. Then there must be a state j which received its lower quota (and $l_j \neq u_j$). That is at one point state i had u_i seats, state j had $\ell \leq l_j$ seats and the Adams method incremented state i, which implies that $\frac{p_i}{f(u_i)} > \frac{p_j}{f(\ell)}$. This leads to

$$A > \frac{p_i}{u_i} = \frac{p_i}{f(u_i)} > \frac{p_j}{f(\ell)} \geq \frac{p_j}{l_j} \geq A,$$

which is a contradiction. □

Sometimes it is more convenient to work with the inverse of the claim function: i.e. with $\frac{f(s)}{p_i}$ rather than with $\frac{p_i}{f(s)}$. In such cases the divisor methods increment the states with the smallest $\frac{f(s)}{p_i}$ value. The following lemma is needed for estimating the departure.

Lemma 5. *For any apportionment problem $(\boldsymbol{p}, H)$ and for any $i, j \in N$, $\frac{l_j}{p_j} \leq \frac{u_i}{p_i}$.*

Proof.

$$\frac{l_j}{p_j} = \frac{\lfloor \frac{p_j}{A} \rfloor}{p_j} \leq \frac{1}{A} \leq \frac{\lceil \frac{p_i}{A} \rceil}{p_i} = \frac{u_i}{p_i}. \quad \square \tag{1}$$

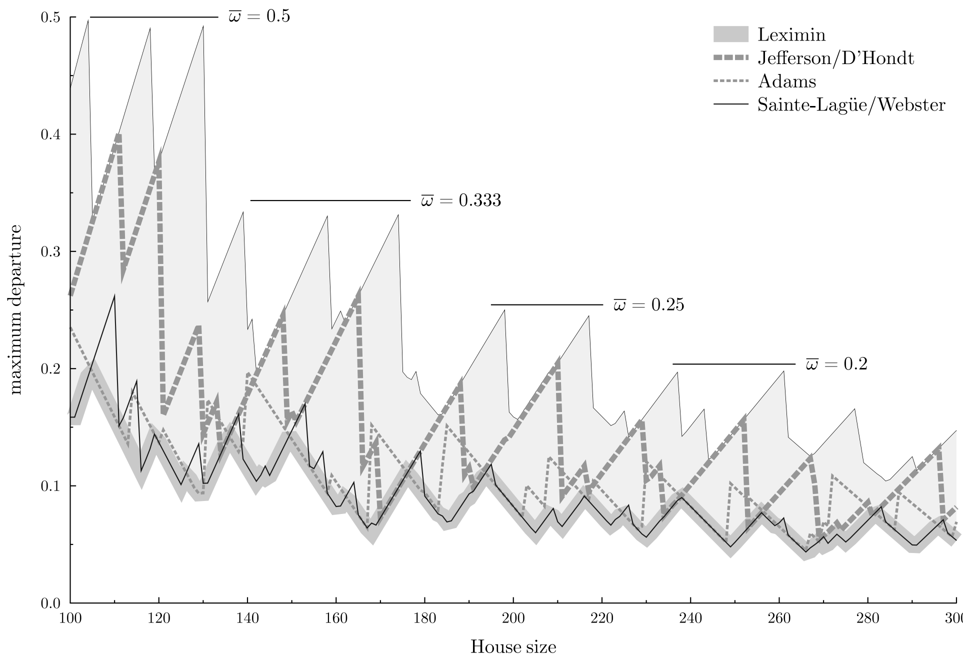

Fig. 1. Apportionment over Norwegian counties. Leximin coincides with β, Saint-Lagüe/Webster is near, Adams is somewhat worse, while the Jefferson/D'Hondt method performs poorly, reaching ω several times.

Regular divisor methods, just like the Leximin method, use large states as buffers to allot or acquire extra seats. The next theorem formulates a weaker statement for regular linear divisor methods, namely, that if the number of seats allotted to a state exceeds its upper (respectively lower) quota, then states which are allotted their lower (upper) quota are necessarily smaller.

Theorem 6. *Let M_f be a regular divisor method with a linear division criterion. If state i receives more seats than its upper quota then each state j which received its lower quota is smaller, that is $p_j < p_i$. If state j receives less seats than its lower quota then each state i which received its upper quota is smaller, that is $p_i < p_j$.*

Proof. By Remark 1 we may assume that $f(s) = s + \nu$, where $\nu \le 1$. If i received more seats than its upper quota, then due to Lemma 2 there must be a state j that received its lower quota, that is $\frac{f(l_j)}{p_j} > \frac{f(u_i)}{p_i}$.

$$\frac{f(l_j)}{p_j} > \frac{f(u_i)}{p_i} \iff \frac{l_j+\nu}{p_j} > \frac{u_i+\nu}{p_i} \overset{\text{Eq. (1)}}{\Longrightarrow}$$

$$\frac{\nu}{p_j} > \frac{\nu}{p_i} \iff p_i > p_j.$$

For the second part, suppose that j received ℓ number of seats which is strictly less than its lower quota. Then

$$\frac{f(l_i)}{p_i} < \frac{f(\ell)}{p_j} \le \frac{f(l_j-1)}{p_j} = \frac{l_j+\nu-1}{p_j}.$$

We may assume that $\nu - 1 < 0$, as $\nu = 1$ yields the Jefferson/D'Hondt method which does not violate the lower quota. If there is a state that received less seats than its lower quota then by Lemma 2 there must be a state i which received its upper quota for which $u_i \neq l_i$, i.e. $l_i = u_i - 1$. Let us suppose that $\nu - 1 < 0$, then

$$\frac{f(l_i)}{p_i} < \frac{f(l_j-1)}{p_j} \iff \frac{l_i+\nu}{p_i} < \frac{l_j+\nu-1}{p_j} \iff$$

$$\frac{u_i+\nu-1}{p_i} < \frac{l_j+\nu-1}{p_j} \overset{\text{Eq.(1)}}{\Longrightarrow} \frac{\nu-1}{p_i} < \frac{\nu-1}{p_j} \iff p_j > p_i. \quad \square$$

Finally, we can state our main result.

Theorem 7. *Let M_f be a regular divisor method with a linear division criterion. Then for any apportionment problem $(\boldsymbol{p}, H)$ and state $i \in N$,*

$$\left|\frac{\frac{p_i}{M_f(\boldsymbol{p},H)_i} - A}{A}\right| \le \omega. \tag{2}$$

Proof. Again by Remark 1 we may assume that $f(s) = s+\nu$, where $\nu \le 1$. Consider an apportionment problem $(\boldsymbol{p}, H)$ where the upper quota is violated by M_f. Due to Lemma 2, the lower quota cannot be violated at the same time.

Let i be a state that received k seats, which is more than its upper quota. By Lemma 2 there is a state j that received its lower quota. That is, at one point i has $k-1$ seats, j has l_j seats, and the divisor method increments i, therefore $\frac{p_i}{f(k-1)} \ge \frac{p_j}{f(l_j)}$. Note that by Theorem 6, $p_i > p_j$. This implies $\frac{p_i}{f(k-1)+1-\nu} \ge \frac{p_i}{f(l_j)+1-\nu}$. Thus

$$\left|\frac{\frac{p_i}{k} - A}{A}\right| = \left|\frac{\frac{p_i}{k-1+\nu+1-\nu} - A}{A}\right| = \left|\frac{\frac{p_i}{f(k-1)+1-\nu} - A}{A}\right| =$$

$$-\left(\frac{\frac{p_i}{f(k-1)+1-\nu} - A}{A}\right) \le -\left(\frac{\frac{p_j}{f(l_j)+1-\nu} - A}{A}\right) =$$

$$-\left(\frac{\frac{p_j}{u_j} - A}{A}\right) = \left|\frac{\frac{p_j}{u_j} - A}{A}\right| \le \omega_j \le \omega.$$

Now let j be a state that received m seats, which is less than its lower quota. By Lemma 2 there is a state i that received its upper quota. Let ℓ be the number of seats j had when i received its upper quota. So, at one point, j had $\ell \le m$ seats, i had l_i seats and the method incremented i, therefore $\frac{p_j}{f(\ell)} \le \frac{p_i}{f(l_i)}$. By Theorem 6 $p_i > p_j$, which in turn implies that $\frac{p_j}{f(\ell)-\nu} \le \frac{p_i}{f(l_i)-\nu}$ for $0 \le \nu < 1$. Thus

$$\left|\frac{\frac{p_j}{m} - A}{A}\right| = \left|\frac{\frac{p_j}{m+\nu-\nu} - A}{A}\right| \le \left|\frac{\frac{p_j}{f(\ell)-\nu} - A}{A}\right| =$$

$$\frac{\frac{p_j}{f(\ell)-\nu}-A}{A} \leq \frac{\frac{p_i}{f(l_i)-\nu}-A}{A} = \frac{\frac{p_i}{l_i}-A}{A} = \left|\frac{\frac{p_i}{l_i}-A}{A}\right| \leq \omega_j \leq \omega.$$

We exclude $\nu = 1$ due to the lack of lower quota violation. □

6. Discussion

Our results have a direct application. Given a regular linear divisor method and a vector of state populations, we can pin down a house size which guarantees that the maximum departure does not exceed a given limit (e.g. the recommendation of the Venice Commission).

For an illustration we look at the Norwegian Parliament: The Storting accommodates 169 seats, which are elected in 19 counties. Seats are distributed according to a modified version of the Sainte-Laguë/Webster method using adjusted population data. When calculating the size of a county, its population is adjusted with its area, and instead of the usual $f(s) = s + 1/2$ divisor they use $f(0) = 0.7$ for $s = 0$. Interestingly, in practice this modification never plays a role.

Aust-Agder, the smallest county in this adjusted sense, gets 4 seats, and its respective share is between 3 and 4 seats for the current house size. In other words, even in the worst case the maximum departure will not exceed 33% (see Fig. 1). The Sainte-Laguë/Webster method performs quite well on the Norwegian data, and the actual departure is much lower, but fluctuations in the population data (e.g. due to migration) are quite common and the method is known to produce allotments on the ω bound (see the Belgian example in [5]). By increasing the house size by a mere 6 seats to 175 the respective share of Aust-Agder is between 4 and 5 seats and $\overline{\omega}$, the upper bound for maximum departure drops to 25%.

Empirical evidence hints that all regular divisor methods (including the Dean and the Huntington–Hill/EP methods) stay within the ω bound. This, however does not solve the case of the USA. In order to reduce the huge gap between the constituencies of Rhode Island and Montana, the size of the House of Representatives would have to be increased to 871, more than twice its current size [2]! Here another approach may be more rewarding: the Leximin method is known to coincide with β most of the time, for which $\overline{\beta}$ composes a – much lower – upper bound.

Where even the Leximin method performs poorly, increasing the size of the administrative regions can greatly decrease the departure. Biró et al. [2] present the case of Hungary, where an apportionment based on regions rather than counties reduces departure from 15.28% to 3.37%. Decreasing the number of administrative regions from 20 to 7 increased l_{sm}, the lower quota of the smallest region, from 2 to 10. The improvement is, therefore, not due to sheer luck as the corresponding $\bar{\omega}$ reduces from 50% to 10% and $\bar{\beta}$ from 20% to less than 5%!

Balinski and Young [1] argued that there is no perfect apportionment method: no method that satisfies Hare-quota, and avoids the Alabama and population paradoxes at the same time. Empirical evidence shows that the Huntington–Hill/EP and the Sainte-Laguë/Webster methods hardly ever violate quota. More importantly they propose allotments, which are usually close to the Leximin solution. For countries where monotonicity issues are a real problem, these methods constitute a good compromise as well.

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